Analysis of TM scattering from open two-dimensional perfectly conducting shells using aggregated compactly-supported wavelet bases

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Abstract: A technique for the efficient analysis of electromagnetic scattering from open conducting structures is presented. The technique is based on the construction of a compactly supported aggregated wavelet basis that is used to transform vectors of arbitrary length. The use of this technique leads to a sparse method of moments matrix, and speeds up matrix-vector products required in the iterative solution. The proposed algorithm can be appended to standard MoM codes and is capable of automatically handling an arbitrary number of unknowns and open-conductor configurations. For closed conducting structures, the method reduces to previously existing wavelet-based techniques.

1 Introduction

Electromagnetic scattering problems are often formulated in terms of integral equations that are solved numerically using the Method of Moments (MoM) [1]. The solution of scattering problems using the MoM requires the inversion of a dense matrix, either by direct or iterative methods. Because of the $O(N^3)$ computational complexity associated with this operation, the electromagnetics community has shown a vigorous interest in fast algorithms for solving large problems. Several techniques have been proposed that permit a fast evaluation of a matrix-vector multiplication for matrices that arise in the MoM-based solution of scattering problems. These techniques may drastically reduce the computational cost associated with the iterative solution of large scattering problems. Examples include, but are not limited to, the Impedance Matrix Localisation (IML) technique [2], the Complex Multipole Beam approach (CMBA) [3], and wavelet based techniques [4–9]. Both the IML and the CMBA rely on forming directive radiation beams by using special current basis and testing functions that render an MoM matrix sparse. However, they are limited in scope to scattering by sufficiently smooth bodies.

Wavelet-based techniques provide a multiresolution description of the surface current density. Wavelets with small spatial support contain higher spatial frequencies than wavelets with larger spatial support, and are poorer radiators. They therefore 'interact' to a lesser extent with other wavelets which, in turn, leads to a sparse MoM matrix. Wavelet techniques have been used for investigating TE and TM scattering from 2-D closed cylinders of arbitrary shape [6–9], for analysing a dihedral reflector [5], for static 2-D problems [10], and for waveguide mode computation [11, 12]. These techniques rely on compactly-supported wavelet bases [13, 14] and on the Fast Wavelet Transform (FWT) [15, 16], an algorithm with linear computational complexity for transforming data sets comprising a number of points that is a power of two.

In this paper, we present a simple wavelet-based formulation to sparsify the MoM matrix describing the scattering from open conducting structures that are modelled by using an arbitrary number of unknowns. For closed structures discretised in terms of an integer power of two unknowns, the proposed technique reduces to the previously-reported methods. The proposed technique is extremely efficient and flexible. It utilises the FWT and Daubechies wavelets [14–16] and can handle an arbitrary number of unknowns and open conductor distributions automatically. This technique can be incorporated into already existing MoM codes as an efficient 'post-processor' constructed around the FWT and the aggregated wavelet bases introduced here.

2 Formulation of 2-D TM scattering

Consider an open conducting shell, with a corresponding connected or disconnected surface C, that resides in free space. The Electric Field Integral Equation (EFIE) relates the surface current distribution $J_z(\rho')$ on C (Fig. 1) to the incident field $E_z^{inc}(\rho)$ on C in the following manner:

$$E_z^{inc}(\rho) = \frac{\beta\eta}{4}\int_C J_z(\rho')H_0^{(2)}(\beta|\rho-\rho'|)dc' \quad \forall\rho\in C \quad (1)$$

where β is the wavenumber, η is the free-space impedance, and $H_0^{(2)}(.)$ is the Hankel function of the second kind of order zero. Eqn. 1 can be solved by using the MoM. This is typically facilitated by expanding the unknown current $J_z(\rho')$ in a pulse basis, and by utilis-

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ing delta testing functions. This procedure yields a dense matrix equation [1] of the form

$$\mathbf{E} = \mathbf{ZI} \tag{2}$$

where the unknown vector **I** corresponds to the coefficients of the pulse basis representation of $J_z(\rho')$, and the vector **E** contains samples of the incident electric field $E_z^{inc}(\rho)$ at a finite number of observation points on C. Eqn. 2 can be solved using direct or iterative techniques. Alternatively, the matrix equation may be formulated by using a wavelet basis and testing set. When compared to the original matrix, many more of the elements of the transformed matrix are negligibly small compared to the large entries such as those residing on the diagonal. The transformed matrix can then safely be rendered sparse by setting to zero all matrix entries whose magnitudes are smaller than a threshold value. The solution is then carried out in the wavelet domain. The sparse nature of the matrix can be exploited to reduce the operation count associated with the solution of the matrix equation.

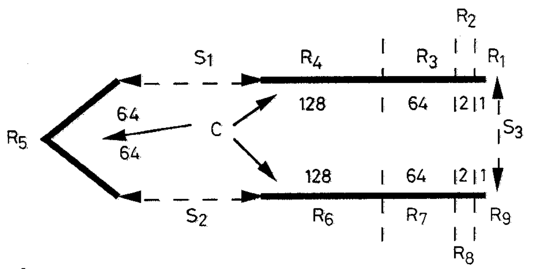

Fig. 1 *Forming the component subsets for 2-D perfectly-conducting objects*
——— perfectly conducting portions, which together form the surface C
◄– – –► regions with no conductors
– – – – boundaries between component subsets of R
The structure consists of three separated conducting regions, modelled by 195, 128, and 195 unknowns. Numbers near conducting regions correspond to the total number of unknowns in each of the conducting sub-sections. Component subsets of R are chosen to be conducting portions with an integer power of two number of unknowns

3 Orthonormal wavelet bases

Wavelet bases, described at length in a variety of excellent texts [13, 14, 17, 18] comprise a class of functions that provides a multiresolution analysis of functions that are square-integrable. This multiresolution analysis is provided through the construction of a function Ψ, termed the prototype wavelet. A discrete square-integrable function f can then be expanded in the following manner:

$$f(x) = \sum_i \sum_j \langle f(x), \Psi_{ij}(x) \rangle \Psi_{ij}(x) \tag{3}$$

where $\langle\ \rangle$ denotes inner product and

$$\Psi_{ij}(x) = 2^{i/2} \Psi(2^i x - j) \tag{4}$$

The double index notation associated with a wavelet Ψ_{ij} is used to specify the scale i and the shift j. Orthonormal wavelets have the property that

$$\langle \Psi_{ij}, \Psi_{kl} \rangle = \delta_{ik}\delta_{jl} \tag{5}$$

where δ denotes the Kronecker delta function.

Orthogonal discrete wavelet bases with compact support have been developed by Daubechies [14] for representing square-integrable signals having a length that is a power of two. These have been used to solve 2-D scattering problems for closed conducting shells [8], for analysing a dihedral reflector [5], and for 2-D static problems [10]. In the following section, aggregated wavelet bases are developed with the intention of solving 2-D scattering problems involving open conducting shells of arbitrary shape that may be modelled in terms of an arbitrary number of unknowns.

4 Transformation of the MoM matrix by an aggregated compactly-supported wavelet basis

In this section, we present a modified wavelet basis that is derived from previously existing compactly-supported wavelet bases. The intention is to utilise the FWT [15, 16] and Daubechies wavelets [14], to obtain an efficient wavelet-based technique for transforming vectors of arbitrary dimension. This technique will then be utilised to analyse TM scattering from open conducting structures described in terms of an arbitrary number of unknowns. The formulation of this technique is most conveniently presented in the mathematical framework introduced next.

Let there be two collections of subsets S and R of the interval (0, 1),

$$S = \bigcup_{i=1}^{n_S} S_i \tag{6}$$

and

$$R = \bigcup_{j=1}^{n_R} R_j \tag{7}$$

satisfying

$$S \cup R = (0, 1) \tag{8}$$

such that each of the component subsets are disjoint,

$$S_i \cap S_k = \emptyset \qquad i, k \in (1, \ldots, n_S),\ i \neq k \tag{9}$$

$$R_j \cap R_l = \emptyset \qquad j, l \in (1, \ldots, n_R),\ j \neq l \tag{10}$$

$$S_i \cap R_j = \emptyset \qquad i \in (1, \ldots, n_S),\ j \in (1, \ldots, n_R) \tag{11}$$

Also, consider a function $f \in L^2$ (0, 1), i.e., it is square integrable over the unit interval, and let it be identically equal to zero on S. Therefore

$$\text{supp}(f) = R \tag{12}$$

where supp(.) specifies the support, i.e., the region over which a function can have non-zero value. For a given f satisfying eqn. 12, there exists another function $g \in L^2(R)$ such that

$$g(x) = f(x) \qquad \forall x \in R \tag{13}$$

and therefore

$$g \uparrow L^2(0, 1) = f \tag{14}$$

and

$$f \downarrow L^2(R) = g \tag{15}$$

where $\uparrow$ is the extension operator, and $\downarrow$ is the restriction operator. An extension operator forces the function to be zero outside its original domain of definition, and a restriction operator constricts the domain of definition of a function. By the manner in which g has been defined, a representation of g in a basis that spans $L^2(R)$ is also a representation of f, by extending each basis function to L^2 (0, 1). The wavelet-based orthogonal multiresolution basis Ψ that is chosen here is defined in the following manner. Let Ψ_T be an orthogonal wavelet basis for $L^2(T)$, where T is some closed subset of (0, 1). Then, Ψ is defined as

$$\Psi \equiv \left\{ \bigcup_{i=1}^{n_R} (\Psi_{R_i} \uparrow R) \right\} \tag{16}$$

Using this basis Ψ, the basis Ψ' for representing f can be written as

$$\Psi' = \{\Psi \uparrow (0,1)\} \bigcup_{j=1}^{n_S} \{\Psi_{S_j} \uparrow (0,1)\} \tag{17}$$

However, since f is identically equal to zero on S, the coefficients of the basis functions corresponding to the bases $\{\Psi_{Sj}\}_{j=1}^{n_S}$ will all be zero. Hence

$$\Psi' = \Psi \uparrow (0,1) \tag{18}$$

and therefore the coefficients of the basis functions corresponding to the bases $\{\Psi_{Rj}\}_{j=1}^{n_R}$ that represent g can be utilised to represent f. To summarise, the function f is represented by an 'aggregated' orthogonal basis, consisting of several orthogonal wavelet bases defined over disjoint closed subsets where the function f is not constrained to be identically zero.

To demonstrate the construction of the required subsets $\{S_i\}_{i=1}^{n_S}$ and $\{R_j\}_{j=1}^{n_R}$ for representing the current on open conducting structures, consider the specific example presented in Fig. 1. The function f, discussed above, represents the unknown electric current distribution on the 2-D scatterer. Any two perfectly-conducting sections that are separated form disjoint subsets of R. Furthermore, when dealing with an arbitrary number of unknowns placed on the periphery of the 2-D scatterer, each component subset defined on conducting portions should be restricted to have a number of unknowns equal to a power of two in order to solve the problem efficiently, as illustrated in Fig. 1. For example, the upper horizontal conducting portion, modelled by a total of 195 unknowns, is divided into four sections containing 1, 2, 64, and 128 unknowns. Portions that are not conducting form component subsets of S and these curve shapes can in fact be specified arbitrarily, since these are required only formally. It should be noted that all the subsets of R are actually defined on the mapping of these 2-D curves to the interval (0, 1) through the scaled arc-length variable. Mapping of closed 2-D cylinders to the real line using the arc-length variable has been utilised in [6, 8, 9].

The advantage of the subset approach in the discrete case is that problems involving an arbitrary number of unknowns can be handled efficiently. Furthermore, the number of unknowns after transformation to the wavelet basis does not increase. Therefore fullest advantage can be taken of the sparsity introduced by the use of a multiresolution orthogonal basis. Moreover, the constraint that the current be zero on nonconducting portions is automatically satisfied.

The overall aggregated wavelet-basis Ψ' used to represent the function f can be associated with a transformation matrix $\mathbf{W}$. Since the component wavelet bases Ψ_T are orthonormal and real, $\mathbf{W}$ is a unitary matrix, i.e., it satisfies

$$\mathbf{W}^{-1} = \mathbf{W}^T \tag{19}$$

The MoM system of eqn. 2 can then be transformed explicitly to yield

$$\tilde{\mathbf{E}} = \tilde{\mathbf{Z}}\tilde{\mathbf{I}} \tag{20}$$

where

$$\tilde{\mathbf{Z}} = \mathbf{W}\mathbf{Z}\mathbf{W}^T \tag{21}$$

$$\tilde{\mathbf{I}} = \mathbf{W}\mathbf{I} \tag{22}$$

$$\tilde{\mathbf{E}} = \mathbf{W}\mathbf{E} \tag{23}$$

The matrix $\mathbf{Z}$ typically has many negligibly small entries compared to the large entries such as those on the diagonal. The matrix can be sparsified by setting to zero all entries below a threshold. This threshold τ was chosen in [4] and utilised in [7] to be

$$\tau = \frac{\varepsilon \|\tilde{\mathbf{Z}}\|_\infty}{N} \tag{24}$$

Here, we define the threshold τ to be

$$\tau = \varepsilon \max(|\tilde{\mathbf{Z}}_{ij}|)_{1 \le i,\, j \le N} \tag{25}$$

where $|\ |_\infty$ denotes the infinity norm, N is the number of unknowns, and ε is a variable parameter.

Since the MoM equations are complex, the real and imaginary portions are transformed independently to obtain a complex transformed system in the wavelet basis.

As mentioned earlier, the iterative procedure of solving the MoM equations is rendered more efficient in the wavelet domain owing to the sparse nature of the thresholded wavelet-transformed MoM matrix. The specific iterative technique used is the biconjugate gradient method [19]. Furthermore, the number of iterations required in the wavelet domain is essentially the same as in the original domain because the wavelet-transformation is unitary and hence preserves matrix eigenvalues.

A current outstanding problem is the direct computation of the sparse thresholded wavelet-transform of the MoM matrix.

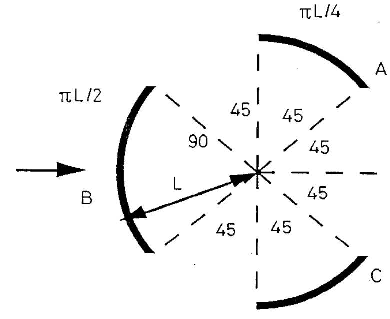

Fig. 2 *Example 1: Representative open-conducting structure consists of three conducting portions (A, B and C) of a circular cylinder, radius L. Lengths, angular locations (in degrees) and a plane wave incident at 180 degrees, are shown*

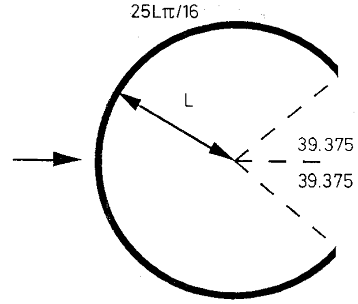

Fig. 3 *Example 2: Representative open-conducting structure consists of a single conducting portion of a circular cylinder, radius L. A plane wave is incident at 180 degrees*

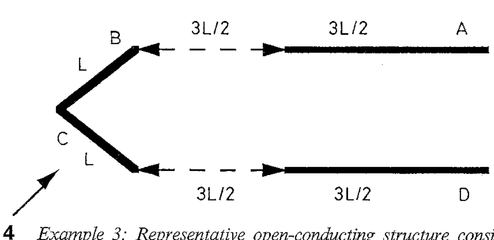

Fig. 4 *Example 3: Representative open-conducting structure consists of four conducting portions (A, B, C and D). A plane wave is incident at an angle of 225 degrees from the section A*

5 Numerical results

The aggregated wavelet-basis technique described in section 4 has been used to analyse TM scattering from several different 2-D open conducting structures. Three representative examples, shown in Figs. 2, 3 and 4, are considered in this section. The first, depicted in Fig. 2, demonstrates the capability of the proposed method to handle structures with separated conducting regions, each of which has a number of unknowns equal to a power of two. The second example (Fig. 3), considers a single open conducting structure modelled using a number of unknowns that is not a power of two. In the third example, shown in Fig. 4, disjointed conducting regions are considered, with each region modelled by a number of unknowns that is not a power of two. These three examples are intended to illustrate the flexibility of the technique in terms of working with an arbitrary number of unknowns, while preserving the accuracy of the computed current distributions and obtaining acceptable sparsity in the transformed MoM matrices.

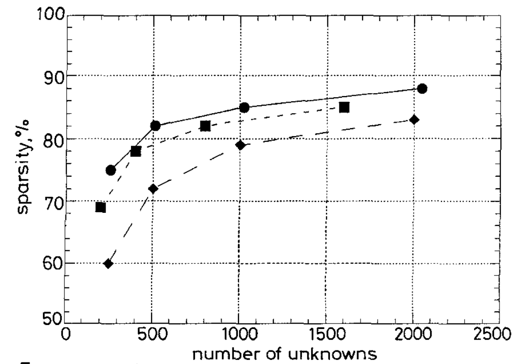

Fig. 5 *Sparsity of wavelet matrices as a function of problem size for the three representative examples*
—●— Example 1
– –■– – Example 2
– –◆– – Example 3

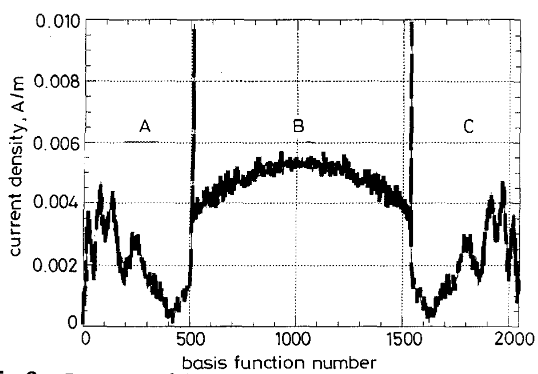

Fig. 6 *Comparison of the current density for example 1 obtained by the standard MoM and the wavelet-based MoM. Total number of unknowns in the case shown is 2048. Conducting portions A, B and C are labelled*
——— MoM
—— — Wavelet-MoM

In each of the examples, the problem size was varied to observe the effect of the number of unknowns on the sparsity of the wavelet-transformed matrix. For examples 1 and 2, L (expressed in terms of free space wavelengths) was taken as $25.6/2\pi$, $51.2/2\pi$, $102.4/2\pi$, and $204.8/2\pi$. For example 3, L took on the values 5, 10, 20, and 40. The corresponding number of unknowns in all cases was kept fixed at ten per wavelength. After transforming the MoM matrix, a thresholding procedure was applied to set to zero all elements in the matrix whose magnitudes fall below the threshold. This threshold was adjusted to get an error of one percent or less in the current density. The resulting sparsity in the transformed MoM matrix is illustrated in Fig. 5. The sparsity increases with problem size, albeit at a decreasing rate, and appears to approach a 'limiting' value for each problem. This limit is related to the fixed relative proportion of wavelets in the 'visible' and 'invisible' regions of the spectrum. For the examples considered, the sparsity is typically between eighty and ninety percent. Figs. 6, 7 and 8 illustrate the excellent accuracy obtained in the current density computed using the wavelet-based MoM technique while maintaining a very good sparsity. For all of these three cases, the threshold parameter ε took the value 5×10^{-4}, and the values taken by L are $204.8/2\pi$, $102.4/2\pi$, and 20, respectively. Figs. 6 and 8 demonstrate that current discontinuities and 'edge-effects' at the ends of the separated conducting regions are also accurately recovered with the aggregated wavelet-basis. For all the cases considered, radar cross-sections obtained using the wavelet bases have been seen to be accurate to within a hundredth of a percent of those computed using the standard bases.

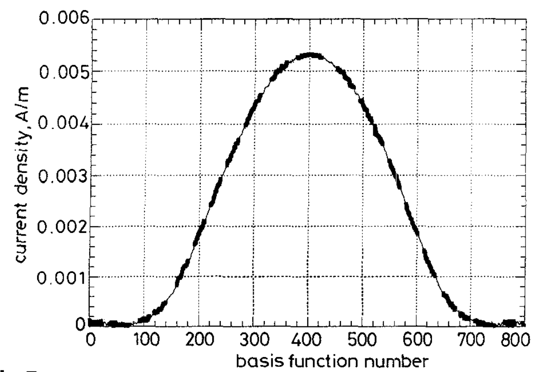

Fig. 7 *Comparison of the current density for example 2 obtained by the standard MoM and the wavelet-based MoM. Total number of unknowns in the case shown is 800*
——— MoM
—— — Wavelet-MoM

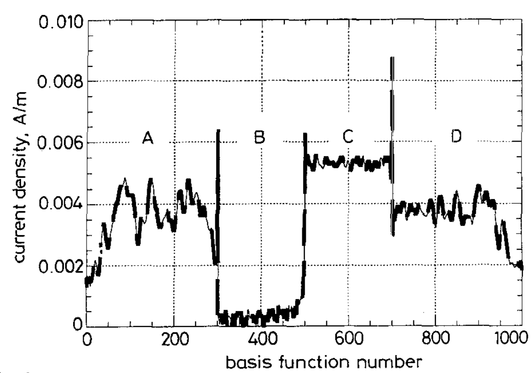

Fig. 8 *Comparison of the current density for example 3 obtained by the standard MoM and the wavelet-based MoM. Total number of unknowns in the case shown is 1000. Conducting portions A, B, C and D are labelled*
——— MoM
—— — Wavelet-MoM

The structure of the thresholded wavelet-transformed MoM matrices for the three examples is shown in Figs. 9, 10 and 11. The matrix for example 1, presented in Fig. 9, consists of three wavelet blocks on the diagonal, each of which has a structure similar to the matrices in [6]. These blocks represent interaction of wavelet functions within each of the three conducting portions comprising example 1. The off-diagonal blocks constitute the interaction between the wavelet functions on separated portions. Similarly, the transformed MoM matrix for example 2 (Fig. 10) contains three wavelet blocks on the diagonal, because the aggregated wavelet basis for this problem consists of three wavelet bases defined on segments, each of which is modelled using a number of unknowns that is a power of two. In Fig. 11, the matrix for example 3 is illustrated. In this case, the matrix possesses less 'structure' because the aggregated wavelet-basis consists of fourteen distinct power-of-two transforms. Small size transforms, say those operating on segments modelled in terms of sixteen or less unknowns typically have a dense structure. This is apparent from the filling observed in some of the diagonal blocks in Fig. 11.

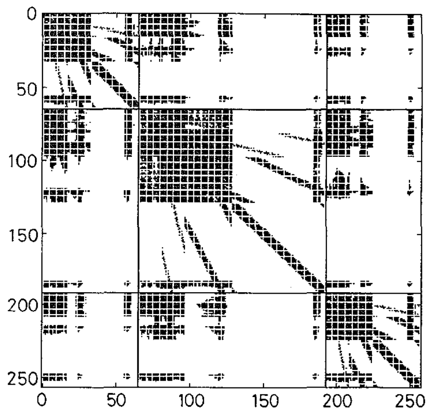

Fig. 9 *Aggregated wavelet-transform of MoM matrix, for example 1*

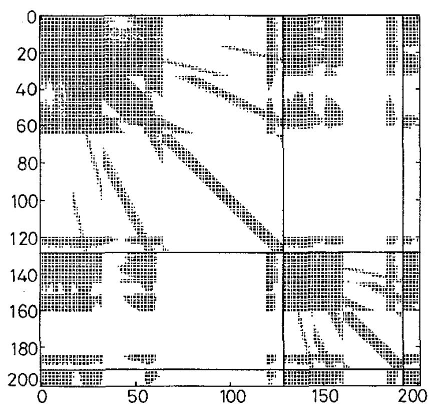

Fig. 10 *Aggregated wavelet-transform of MoM matrix, for example 2*

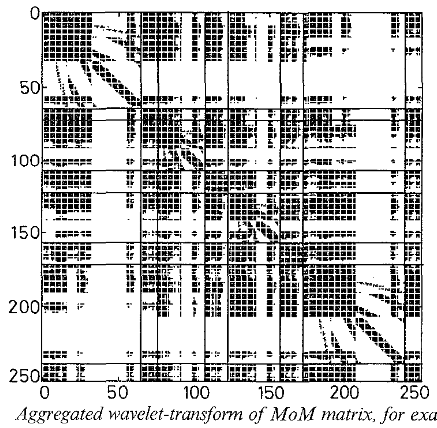

Fig. 11 *Aggregated wavelet-transform of MoM matrix, for example 3*

6 Conclusions

The aggregated compactly-supported basis described in this paper provides a rigorous and efficient means for analysing TM scattering from arbitrarily shaped 2-D open and closed scatterers with an arbitrary number of unknowns. Typically, eighty to ninety percent sparsity has been observed in the wavelet-transformed MoM matrices, when the current density error was less than one percent. The aggregated wavelet-basis technique can easily be appended to already existing MoM codes. An extension to the TE case is straightforward.

The technique described in this paper could be extended to incorporate 2-D planar wavelet bases in order to efficiently analyse scattering from a larger class of structures. This extension is currently being examined.

In this paper, the Daubechies wavelets [14] have been used to construct the aggregated wavelet bases. These wavelets have some 'edge-effects' which can be alleviated by using spline wavelets [12] or intervallic wavelets [9, 20]. Aggregated wavelet bases derived from such wavelets could lead to sparser matrices.

7 References

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