A Generalized TDIE Framework for Arbitrary Time Basis Functions

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A generalized formulation of the TD-EFIE is presented, extending the work of Chung *et al.*, [1]. The framework shows that, for a large class of basis functions, the time integration can be analytically removed from the calculation. This allows the time-space integral to be computed to arbitrary accuracy through existing quadrature schemes. The result is a single, large matrix problem that is adaptable to different solvers, including marching on in time, marching on in order, fast solvers, and parallel solvers.

1 Introduction

The time-domain integral equations (TDIEs) provide a powerful tool for efficiently solving broadband electromagnetic problems. Transient analysis allows the computation of responses over a wide of frequencies using a single simulation, and the surface-based formulation can provide a more efficient solution than a volumetric approach in many cases. The standard formulation involves a marching on in time (MOT) scheme, where the unknowns at the current time step are dependent on the unknowns at previous time steps. A new formulation was developed by Chung *et al.*, [1], where weighted, entire-domain Laguerre polynomials are used for time basis functions, and a time testing operation is introduced. The time testing reduces the time-dependent basis functions to functions dependent only on delay between points on the surface, a spatial computation. Using weighted Laguerre polynomial temporal basis functions in this manner provides a more stable, more computationally efficient solution. A formulation follows where the solution uses a marching on in order scheme. Unknowns associated with a given Laguerre polynomial depends on polynomials of lower order. A marked improvement in stability was demonstrated for both the PEC and dielectric cases. However, weighted Laguerre polynomials may not be ideal basis functions for all cases.

The properties of the formulation in [1] are not unique to weighted Laguerre polynomials; they are actually a special case of a more generalized formulation. Through temporal testing, temporal dependence can be analytically removed using almost any time basis function. This will increase the stability of TDIE codes, and allow the use of situation-appropriate temporal basis functions. This is important for uses such as hybrid FEM/BEM solvers, where a MOT scheme is necessary for the FEM portion, and cases where hierarchical basis functions are appropriate, such as a wavelet time basis. The variety of temporal basis functions will also allow for fast solvers to be used in some cases, where the basis function choice leads to an appropriate LHS matrix

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Though the formulation is given for PEC only, it is equally applicable to other TDIE formulations.

2 Formulation

On a PEC, the Electric Field Integral Equation (EFIE) enforces the boundary condition

$$\hat{\mathbf{n}} \times (\mathbf{E}_s(\mathbf{r}, t) + \mathbf{E}_{inc}(\mathbf{r}, t)) = 0 \tag{1}$$

The scattered electric field, $\mathbf{E}_s(\mathbf{r}, t)$, can be expressed as

$$\begin{aligned} &\mathbf{E}_s(\mathbf{r}, t) = \\ &\quad -\mu \int_S \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\ddot{\mathbf{P}}(\mathbf{r}', t-\tau)\, ds - \nabla\epsilon^{-1} \int_S \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\nabla' \cdot \mathbf{P}(\mathbf{r}', t-\tau)\, ds \end{aligned} \tag{2}$$

where $\mathbf{P}(\mathbf{r}, t)$ is the Hertz vector, $\tau = \frac{|\mathbf{r}-\mathbf{r}'|}{c}$ is the time retardation, and S is the surface of the PEC. Substituting (2) into (1) results in

$$\begin{aligned} \hat{\mathbf{n}} \times \left[\mu \int_S \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\ddot{\mathbf{P}}(\mathbf{r}', t-\tau)\, ds + \nabla\epsilon^{-1} \int_S \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\nabla' \cdot \mathbf{P}(\mathbf{r}', t-\tau)\, ds\right] \\ = \hat{\mathbf{n}} \times \mathbf{E}_{inc}(\mathbf{r}, t) \end{aligned} \tag{3}$$

The Helmholtz vector is approximated by a set of temporal and spatial basis functions as

$$\mathbf{P}(\mathbf{r}, t) = \sum_{n=1}^{N}\sum_{j=1}^{J} P_n^j \mathbf{P}_n(\mathbf{r}) P^j(t) \tag{4}$$

representing a total of NJ unknowns, P_n^j.

To extract test/source interactions, an inner product is taken with respect to temporal and spatial testing functions, $\hat{\mathbf{n}} \times \mathbf{T}_m(\mathbf{r})T^i(t)$, which is the tangential E-field at the testing surface. The integral of the inner product is taken over the domain of the spatial testing function, S_m, and the domain of the temporal testing function, T^i. The inner product of the testing functions with (3) is given by

$$\begin{aligned} \int_{S_m}\int_{T^i} \hat{\mathbf{n}} \times \mathbf{T}_m(\mathbf{r})T^i(t) \cdot \hat{\mathbf{n}} \times \Bigg[\mu \int_{S_n'} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\mathbf{P}_n(\mathbf{r}')\ddot{P}^j(t-\tau)\, ds' \\ +\nabla\epsilon^{-1} \int_{S_n'} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\nabla' \cdot \mathbf{P}_n(\mathbf{r}')P^j(t-\tau)\, ds'\Bigg]\, dt\, ds \\ = \langle \hat{\mathbf{n}} \times \mathbf{T}_m(\mathbf{r})T^i(t), \mathbf{E}_{inc}(\mathbf{r}, t)\rangle_{S_m, T^i} \end{aligned} \tag{5}$$

where S_n' is the domain of the spatial source function. With the condition that $\mathbf{P}_n(\mathbf{r})$ is tangential to the surface, (5) can be rewritten as

$$\begin{aligned} \int_{S_m}\int_{T^i} \mathbf{T}_m(\mathbf{r})T^i(t) \cdot \Bigg[\mu \int_{S_n'} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\mathbf{P}_n(\mathbf{r}')\ddot{P}^j(t-\tau)\, ds' \\ +\nabla\epsilon^{-1} \int_{S_n'} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r}, \mathbf{r}')\nabla' \cdot \mathbf{P}_n(\mathbf{r}')P^j(t-\tau)\, ds'\Bigg]\, dt\, ds \\ = \langle \hat{\mathbf{n}} \times \mathbf{T}_m(\mathbf{r})T^i(t), \mathbf{E}_{inc}(\mathbf{r}, t)\rangle_{S_m, T^i} \end{aligned} \tag{6}$$

Moving the time integral inside leads to

$$\int_{S_m} \mathbf{T}_m(\mathbf{r}) \cdot \left[\mu \int_{S'_n} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r},\mathbf{r}') \mathbf{P}_n(\mathbf{r}') \int_{T^i} T^i(t) \ddot{P}^j(t-\tau)\, dt\, ds' \right.$$
$$\left. + \nabla \epsilon^{-1} \int_{S'_n} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r},\mathbf{r}') \nabla' \cdot \mathbf{P}_n(\mathbf{r}') \int_{T^i} T^i(t) P^j(t-\tau)\, dt\, ds' \right] ds \tag{7}$$
$$= \langle \hat{\mathbf{n}} \times \mathbf{T}_m(\mathbf{r}) T^i(t), \mathbf{E}_{inc}(\mathbf{r},t) \rangle_{S_m,T^i}$$

For a large class of $T^i(t)$ and $P^j(t)$, the time integrals can be evaluated analytically as

$$L^{ij}(\tau) = \int_{T^i} T^i(t) \ddot{P}^j(t-\tau)\, dt \tag{8a}$$

$$K^{ij}(\tau) = \int_{T^i} T^i(t) P^j(t-\tau)\, dt \tag{8b}$$

giving

$$\int_{S_m} \mathbf{T_m}(\mathbf{r}) \cdot \left[\mu \int_{S'_n} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r},\mathbf{r}') \mathbf{P}_n(\mathbf{r}') L^{ij}(\tau)\, ds' \right.$$
$$\left. + \nabla \epsilon^{-1} \int_{S'_n} P_n^j \overline{\overline{\mathbf{G}}}(\mathbf{r},\mathbf{r}') \nabla' \cdot \mathbf{P}_n(\mathbf{r}') K^{ij}(\tau)\, ds' \right] ds \tag{9}$$
$$= \langle \hat{\mathbf{n}} \times \mathbf{T}_m(\mathbf{r}) T^i(t), \mathbf{E}_{inc}(\mathbf{r},t) \rangle_{S_m,T^i}$$

Note that, except for the excitation, this is purely a function of spatial variables. While time has been removed from the equation, the delay, $\tau = \frac{|\mathbf{r}-\mathbf{r}'|}{c}$, remains, which is a function of spatial variables. This allows the use of spatial quadrature techniques [2] capture the space-time integrals represented by $L^{ij}(\tau)$ and $K^{ij}(\tau)$ to arbitrary accuracy. The $L^{ij}(\tau)$ and $K^{ij}(\tau)$ operators represent a 1-D integral, and are easily defined for a variety of basis functions including wavelets, weighted Laguerre polynomials, and the sequential basis functions used in time-stepping schemes (provided they are second order or greater).

In general, this leads to a single, very large matrix solve, where the LHS matrix is size $NJ \times NJ$. The choice of temporal testing and source basis functions will give this matrix certain properties. For example, if the weighted Laguerre polynomial formulation of [1] is used, the matrix will be block-lower triangular. This can be solved with the marching on in order solver in [1], or as a single, large matrix solve. Similarly, if the $T^i(t)$ and $P^j(t)$ are sequential, the matrix will be block diagonal, where all the blocks are sparse. Note also that if $T^i(t) = \delta(t^i)$, we get the standard TDIE formulation back. Again, this can be solved with as MOT or as a single matrix problem. If brute force methods are used, solving the previous two systems with a marching-on scheme has advantages in terms of memory and solution speed. However, the application of fast and parallel solvers to the large matrix formulation could give it an edge over marching-on solvers. The large matrix solve also has advantages in its generality, allowing the easy substitution of appropriate temporal basis functions for different situations.

3 Conclusion

A formulation has been presented by which nearly arbitrary temporal basis functions can be used in TDIEs with very little effort. While the formulation presented is specifically for the EFIE, it is equally applicable to other TDIEs. The formulation removes time from the calculations, allowing the space-time integral of a TDIE to be computed to arbitrary accuracy through existing spatial quadrature techniques. The result is a single, very large matrix solve, which can be redistributed to a multiple-solve scheme in special cases such as MOT-type and weighted Laguerre temporal basis functions. However, the generalized formulation is well suited to research in fast and parallel solvers. The generalized code architecture and numerical results will be discussed at the conference.

References

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